Definitions:

- An ***Experiment***: is a repeatable process that gives rise to a number of *outcomes*
- An ***Event***: is a collection of one or more *outcomes*
- A ***Sample Space***: is the set of all possible outcomes

Example 1:

Two fair spinners each have four sectors numbered 1 to 4. The two spinners are spun together and the sum of the numbers indicated on each spinner is recorded.

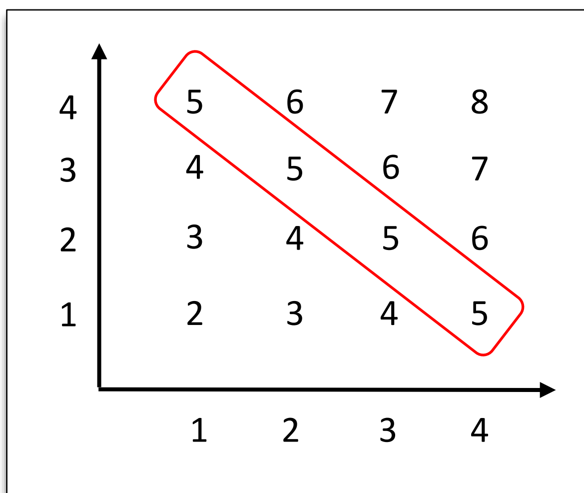
Find the probability of the spinners indicating a sum of:

a exactly 5 **b** more than 5.

We can use a net to show this

a. $P(5) = \frac{4}{16} = \frac{1}{4}$

b. $P(\text{More than } 5) = \frac{6}{16} = \frac{3}{8}$



Example 2:

The table shows the times taken, in minutes, for a group of students to complete a number puzzle.

Time, t (min)	$5 \leqslant t < 7$	$7 \leqslant t < 9$	$9 \leqslant t < 11$	$11 \leqslant t < 13$	$13 \leqslant t < 15$
Frequency	6	13	12	5	4

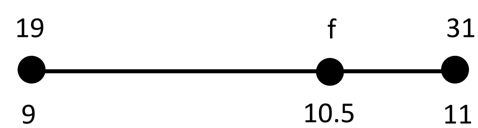
Find the probability that a randomly selected student finished the number puzzle:

a in under 9 minutes **b** in over 10.5 minutes.

Total number of students = 6 + 13 + 12 + 5 + 4 = 40

a. P(under 9 minutes) = $\frac{6+13}{40} = \frac{19}{40}$

b. For this we use interpolation

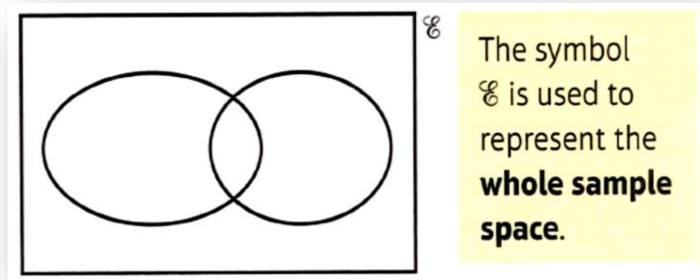


$$\frac{f-19}{31-19} = \frac{10.5-9}{11-9}$$ f = 28

P(over 10.5) = $\frac{40-28}{40} = \frac{12}{40} = \frac{3}{10}$

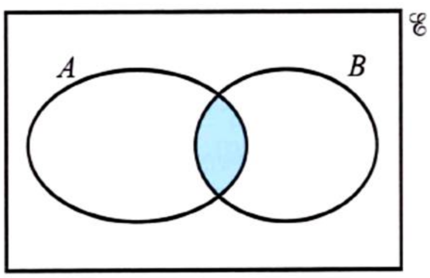
Venn Diagrams:

The rectangle represents the *sample space* while the ellipses represent the events



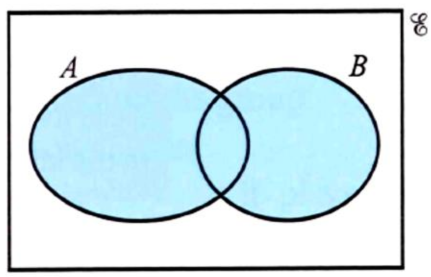
Some famous shapes of the Venn Diagram:

1 The event A **and** B



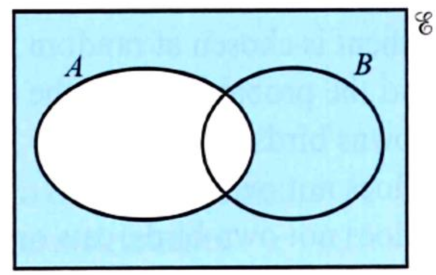
This event is also called the **intersection** of A and B. It represents the event that both A and B occur.

2 The event A **or** B



This event is also called the **union** of A and B. It represents the event that either A or B, or both, occur.

3 The event **not** A



This event is also called the **complement** of A. It represents the event that A does not occur.
$P(\text{not } A) = 1 - P(A)$

Example 1:

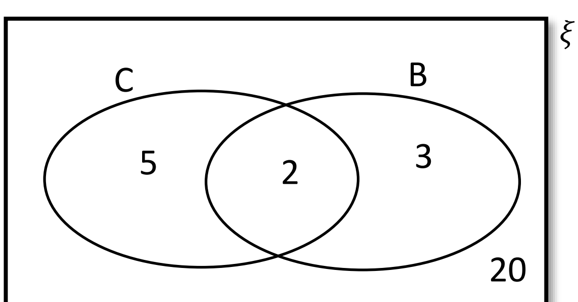
In a class of 30 students, 7 are in the orchestra club, 5 are in the band, and 2 are in both the orchestra club and the band. A student is chosen at random from the class.

a Draw a Venn diagram to represent this information.

b Find the probability that:

i the student is not in the band **ii** the student is not in the orchestra club or the band.

a.



b. i. $P(\text{Not in band}) = \frac{25}{30} = \frac{5}{6}$

ii. $P(\text{Not in club or band}) = \frac{20}{30} = \frac{2}{3}$

Example 2:

A vet surveys 100 of her clients. She finds that:

25 own birds
15 own birds and cats
11 own birds and fish
53 own cats
10 own cats and fish
7 own birds, cats and fish
40 own fish

A client is chosen at random.
Find the probability that the client:

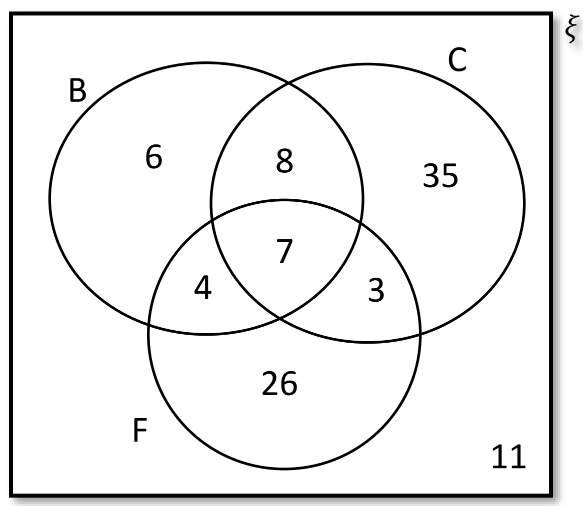
a owns birds only
b does not own fish
c does not own birds, cats or fish.

a. P (birds only) $= \frac{6}{100} = \frac{3}{50}$

b. P (does not own fish) $= \frac{60}{100} = \frac{3}{5}$

c. P (does not own birds, cats or fish)

$= \frac{11}{100}$

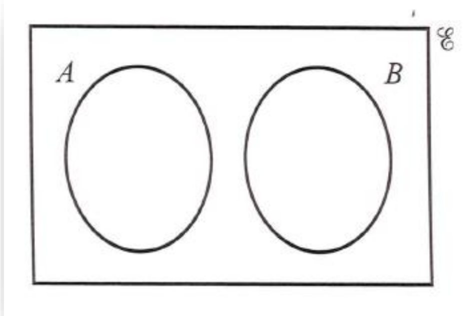


Mutually exclusive and independent events:

When events have no common outcomes, we call them *mutually exclusive*. This shows on a Venn Diagram as no intersection of events.

For mutually exclusive events

$$P(A\ or\ B) = P(A) + P(B)$$



When events have no effect on one another, we call them independent. Which means that the probability of event *A* happening is the same whether event *B* happens or not.

For independent events

$$P(A\ and\ B) = P(A) \times P(B)$$

Important: This rule can be used in reverse. If $P(A\ and\ B) = P(A) \times P(B)$ then *A* and *B* are independent events

Example 1:

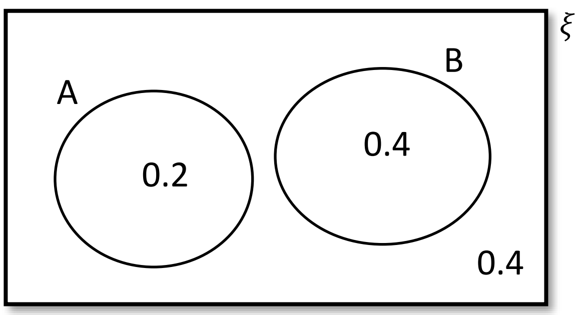
Events *A* and *B* are mutually exclusive events, where P(*A*) = 0.2 and P(*B*) = 0.4

Find: **a** P(*A* or *B*) **b** P(*A* but not *B*) **c** P(neither *A* nor *B*)

a. $P(A\ or\ B) = 0.2 + 0.4 = 0.6$

b. $P(A\ but\ not\ B) = 0.2$

c. $P(neither\ A\ nor\ B) = 0.4$



Example 2:

Events A and B are independent events, where $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$

Find $P(A \text{ and } B)$.

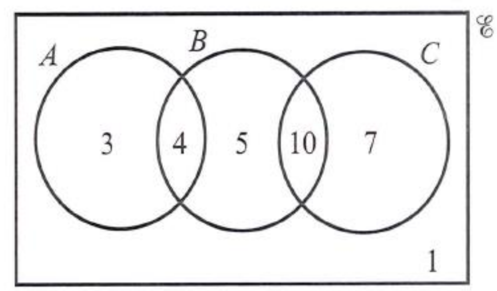
$$P(A\ and\ B) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

Example 3:

The Venn diagram shows the number of students in a class who watch any of three popular TV programmes, A, B and C.

a Find the probability that a student chosen at random watches B or C or both.

b Determine whether watching A and watching B are independent events.



Total number of students = 3+4+5+10+7+1

= 30

a. $P(\text{student watches B or C or both}) = \frac{26}{30} = \frac{13}{15}$

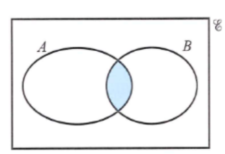
b. $P(\text{A and B}) = \frac{4}{30} = \frac{2}{15}$, $P(A) \times P(B) = \frac{7}{30} \times \frac{19}{30} = \frac{133}{900}$

P(A and B) ≠ P(A) x P(B) so *A* and *B* are not independent events

Set notation:

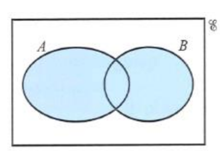
Intersection

A and B ⟶ A ∩ B



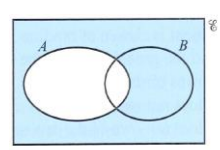
Union

A or B ⟶ A ∪ B



Complement

Not A ⟶ A′ = 1 − P (A)



n(A) = number of elements in event A

Example 4:

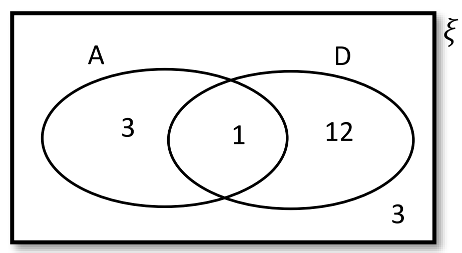
A card is selected at random from a pack of 52 playing cards. Let A be the event that the card is an Ace and D the event that the card is a diamond. Find:

a $P(A \cap D)$ **b** $P(A \cup D)$ **c** $P(A')$ **d** $P(A' \cap D)$

There are 4 Aces in a pack of cards and there are 4 suits in a pack of cards

a. Remember that ∩ means "and" so there is one card in the pack that is an ace of diamond $P(A \cap D) = \frac{1}{52}$

b. Remember that ∪ means "or" so we look for any card that is an ace or a diamond but since there is one card that is both, it shouldn't be counted twice. The Venn diagram helps here. $P(A \cup D) = \frac{16}{52} = \frac{4}{13}$



c. $P(A') = 1 - P(A) = 1 - \frac{4}{52} = \frac{12}{13}$

d. Not an ace and is a diamond $\frac{12}{52} = \frac{3}{13}$

Example 5:

a Given that P(A) = 0.3, P(B) = 0.4 and P($A \cap B$) = 0.25, explain why events A and B are not independent.

b Given also that P(C) = 0.2, that events A and C are mutually exclusive and that events B and C are independent, draw a Venn diagram to illustrate the events A, B and C, showing the probabilities for each region.

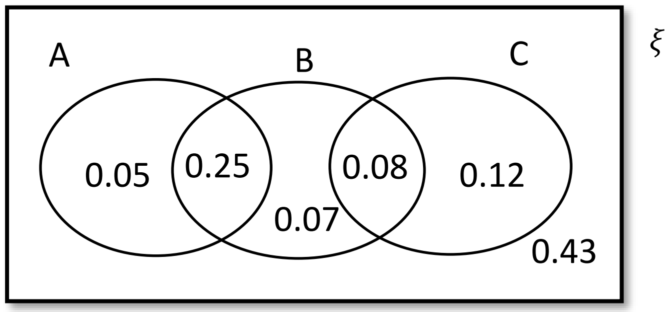
c Find P(($A \cap B'$) $\cup$ C)

a. P(A) x P(B) = 0.3 x 0.4 = 0.12   P(A ∩ B) ≠ P(A) x P(B)

so *A* and *B* are not independent events

b. B and C are independent

P(B ∩ C) = P(B) x P(C) = 0.08



c. P(A ∩ B') = 0.05

P((A ∩ B') ∪ C) = 0.05 + 0.2 = 0.25

A general probability rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 6:

A and B are two events, with P(A) = 0.6, P(B) = 0.7 and P(A ∪ B) = 0.9

Find P(A ∩ B)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ 0.9 = 0.6 + 0.7 - P(A ∩ B)

P(A ∩ B) = 0.4

Conditional Probability

The probability that B occurs given that A has already occurred is written as $P(B \mid A)$

The probability that B occurs given that A has not occurred is written as $P(B \mid A')$

For independent events $P(B \mid A) = P(B \mid A') = P(B)$ This can be used to check independence of events

Example 1:

A school has 75 students in year 12. Of these students, 25 study only humanities subjects (H) and 37 study only science subjects (S). 11 students study both science and humanities subjects.

a Draw a two-way table to show this information.

b Find:

i $P(S' \cap H')$ **ii** $P(S|H)$ **iii** $P(H|S')$

a.

	H	H'	Σ
S	11	37	48
S'	25	2	27
Σ	36	39	75

b. i. $P(S' \cap H') = \frac{2}{75}$

ii. $P(S \mid H) = \frac{11}{36}$ (Remember that event H has already happened which means that we are in the given 36 students only)

iii. $P(H \mid S') = \frac{25}{27}$ (Remember that event S' has already happened which means that we are in the given 27 students only)

Example 2:

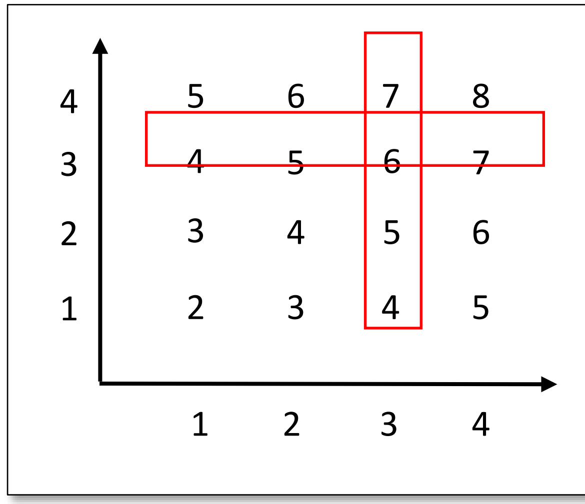
Two four-sided dice are thrown together, and the sum of the numbers shown is recorded.

a Draw a sample-space diagram showing the possible outcomes.

b Given that at least one dice lands on a 3, find the probability that the sum on the two dice is exactly 5.

c State one modelling assumption used in your calculations.

a.



b. At least one dice lands on 3 happens 7 times, of those 2 times give a sum of 5

$P(5|3) = \frac{2}{7}$

c. Dice are fair so all possible outcomes are equally likely

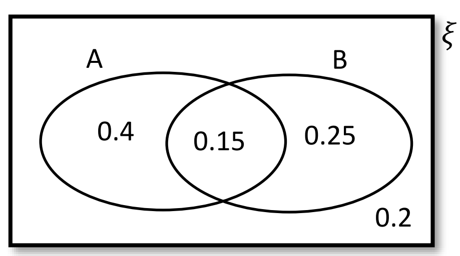
Example 3:

A and B are two events such that $P(A) = 0.55$, $P(B) = 0.4$ and $P(A \cap B) = 0.15$

a Draw a Venn diagram showing the probabilities for events A and B.

b Find:

i $P(A | B)$ **ii** $P(B | (A \cup B))$ **iii** $P(A' | B')$



a. $P(A|B) = \frac{0.15}{0.4} = \frac{3}{8}$

b. $P(B|(A \cup B)) = \frac{0.4}{0.4+0.15+0.25} = \frac{1}{2}$

c. $P(A'|B') = \frac{0.2}{0.4+0.2} = \frac{1}{3}$ (Remember we are working only in B’)

A general conditional probability rule:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

Exercise:

C and D are two events such that $\mathrm{P}(C) = 0.2$, $\mathrm{P}(D) = 0.6$ and $\mathrm{P}(C \mid D) = 0.3$

Find:

a $\mathrm{P}(C \cap D)$ **b** $\mathrm{P}(D \mid C)$ **c** $\mathrm{P}(C \cup D)$

a. $P(C \mid D) = \frac{P(C \cap D)}{P(D)}$ $0.3 = \frac{P(C \cap D)}{0.6}$ $P(C \cap D) = 0.18$

b. $(D \mid C) = \frac{P(C \cap D)}{P(C)} = \frac{0.18}{0.2} = 0.9$

c. $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 0.2 + 0.6 - 0.18 = 0.62$

Tree diagrams

This type of diagrams helps with outcomes of several events happening in succession

Example 1:(Simple Probability)

A bag contains seven green beads and five blue beads. A bead is taken from the bag at random and not replaced. A second bead is then taken from the bag.

Find the probability that:

a both beads are green

b the beads are different colours.

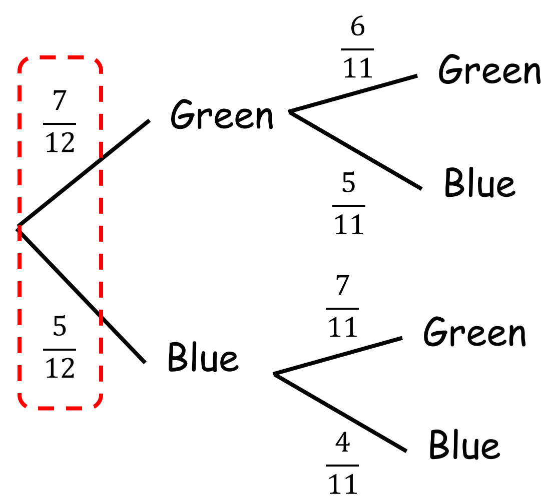
First we draw a probability tree diagram. Total beads = 12

Remember that "*and is* $\times$" where "*or is* $+$"

a. P(Both beads are green)

$$= \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$$

Total probability = 1

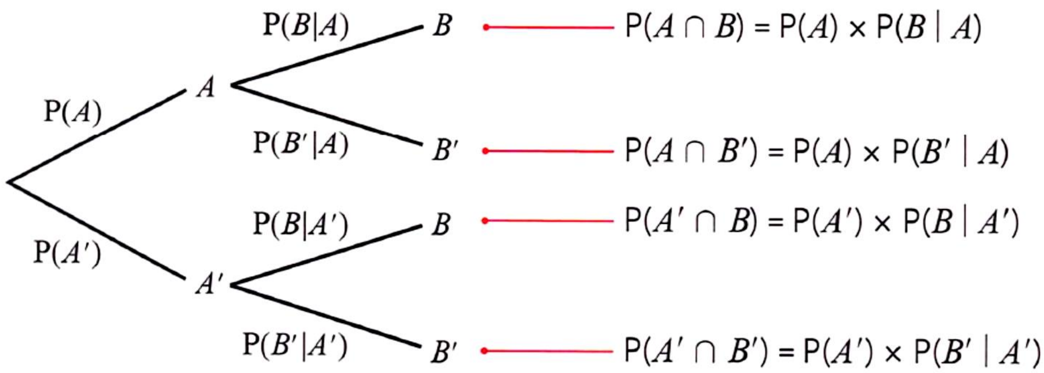


b. P(Both beads are different colours)

$$= \frac{7}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{7}{11} = \frac{35}{66}$$

(Green and Blue) or (Blue and Green)

Tree diagrams for *Conditional Probability*:



It is very important here to think in terms of "*Conditional*"

Example 2:(Conditional Probability)

The turnout of spectators at a Formula 1 race is dependent upon the weather. On a rainy day, the probability of a big turnout is 0.4, but if it doesn't rain, the probability of a big turnout increases to 0.9. The weather forecast gives a probability of 0.75 that it will rain on the day of the race.

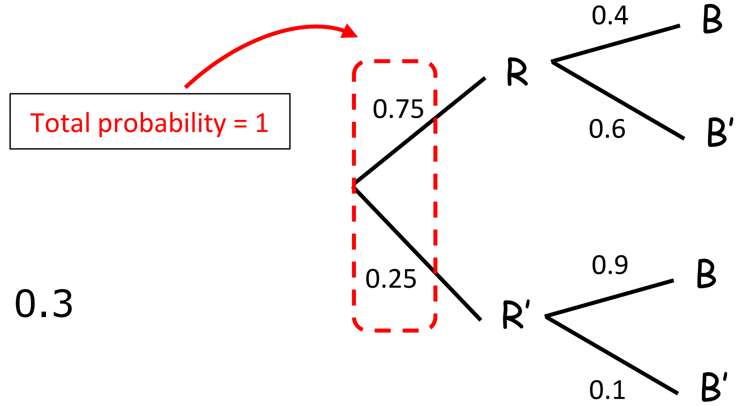
a Draw a tree diagram to represent this information.

Find the probability that:

b there is a big turnout and it rains

c there is a big turnout.

a. Let's choose our letters first. R means that it's going to rain and B means we have a Big turnout



b. P(B and R) = 0.4 x 0.75 = 0.3

c. P(B) 0.3 + 0.25 x 0.9 = 0.525

Example 3:(Conditional Probability)

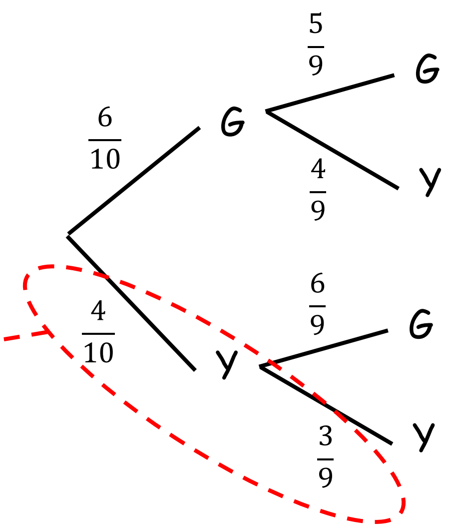
A bag contains 6 green beads and 4 yellow beads. A bead is taken from the bag at random, the colour is recorded and it is not replaced. A second bead is then taken from the bag and its colour recorded. Given that both beads are the same colour, find the probability that they are both yellow.

The problem says "Given that both beads are the same colour" before saying "they are both yellow" so this is conditional probability.

P(Both yellow | Both are the same colour) =

And

$$= \frac{\text{Both are yellow} \cap \text{Both are same colour}}{\text{Both are same colour}}$$



$$= \frac{\frac{4}{10}\times\frac{3}{9}}{\frac{6}{10}\times\frac{5}{9}+\frac{4}{10}\times\frac{3}{9}} = \frac{2}{7}$$

Both are green or both are yellow