Discrete Random Variable: Is a variable whose value depends on the outcome of a *random* event but can only take *certain* numeric values (Discrete)

Example 1:

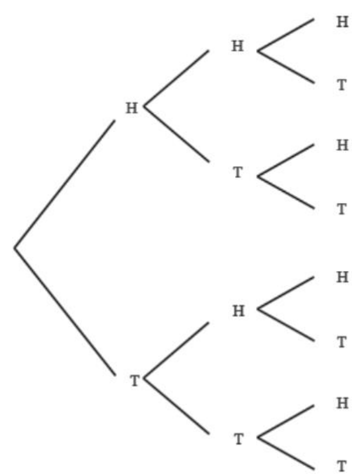
Three fair coins are tossed.

a Write down all the possible outcomes when the three coins are tossed.

A random variable, X, is defined as the number of heads that appear when the three coins are tossed.

b Write the probability distribution of X as:

i a table **ii** a probability function.



a. {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

b. i.

No. of Heads (x)	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

ii. $P(X = x) = \begin{cases} \frac{1}{8} & x = 0, 3 \\ \frac{3}{8} & x = 1, 2 \end{cases}$

Example 2:

A biased four-sided dice with faces numbered 1, 2, 3 and 4 is rolled. The number on the bottom-most face is modelled as a random variable X.

a Given that $P(X = x) = \frac{k}{x}$, find the value of k.

b Write the probability distribution of X in table form.

c Find the probability that:

i $X > 2$ **ii** $1 < X < 4$ **iii** $X > 4$

a. Sum of all probabilities = 1 and since the model is $\frac{k}{x}$ then

$$\frac{k}{1}+\frac{k}{2}+\frac{k}{3}+\frac{k}{4} = 1 \quad \therefore \quad k\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = 1$$

$$\therefore k\left(\frac{25}{12}\right) = 1 \qquad \therefore k = \frac{12}{25}$$

b. Using $\frac{k}{x}$ with $k = \frac{12}{25}$ we get

x	1	2	3	4
$P(X = x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

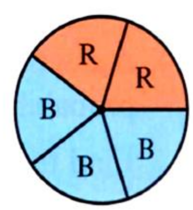
c. i. $P(X > 2) = \frac{4}{25} + \frac{3}{25} = \frac{7}{25}$

ii. $1 < P(X) < 4 = \frac{6}{25} + \frac{4}{25} = \frac{2}{5}$

iii. $P(X > 4) = 0$

Example 3:

A fair spinner is spun until it lands on red or has been spun four times in total. Find the probability distribution of the random variable S, the number of times the spinner is spun.



Here we have 2 conditions. Either the spinner lands on *Red* or has been spun *4 times*, and we should take into consideration the details of how this works. For example, if you need to spin 2 times, this means that it did not land on Red the first time (landed on blue) and so on

$P(S=1) = \frac{2}{5}$ $\qquad$ $P(S=2) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

$P(S=3) = \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{18}{125}$

For S = 4 there is something important to note: The experiment might yield 3 times landing on *Blue* then landing on *Red* OR Not landing on Red at all but we reached the 4 times barrier.

So If we go the traditional way it will look like this

$P(S=4) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

An *easier approach* is to remember that the sum of all probabilities is 1

So P(S=4) = 1 – all past probabilities

$= 1 - \left(\frac{2}{5} + \frac{6}{25} + \frac{18}{125}\right) = \frac{27}{125}$

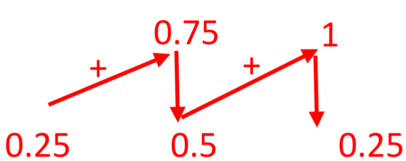
Cumulative distribution function for a discrete random variable

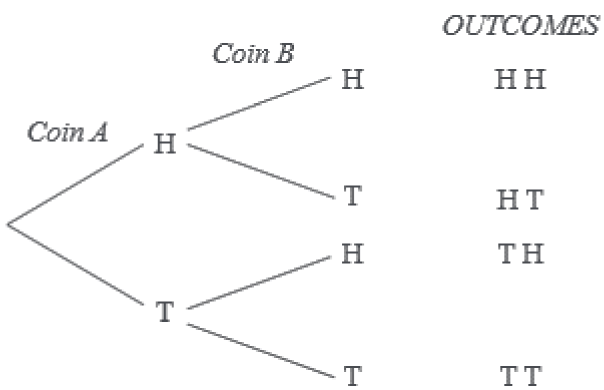
This function is called $F(x)$ where $F(x) = P(X \leq x)$

Example 1:

Two fair coins are tossed. X is the number of heads showing on the two coins.
Draw a table to show the cumulative distribution function for X.

x	0	1	2
$P(X = x)$	0.25	0.5	0.25
$F(x)$	0.25	0.75	1





Example 2:

The discrete random variable X has a cumulative distribution function $F(x)$ defined by:

$$F(x) = \frac{(x+k)}{8}; \quad x = 1, 2, 3$$

a Find the value of k.

b Draw the distribution table for the cumulative distribution function.

c Write down F(2.6)

d Find the probability distribution of X.

a. The maximum of the cumulative function is 1 and happens at $x = 3$

so $\frac{3+k}{8} = 1$ k = 5

b.

x	1	2	3
$F(x)$	$\frac{3}{4}$	$\frac{7}{8}$	1

c. This is a discrete variable so there is no 2.6 Less than 2.6 is actually

less than 2 so $F(2.6) = F(2) = \frac{7}{8}$

d. From the cumulative we can work in reverse

P(1) = F(1) = $\frac{3}{4}$ P(2) = $\frac{7}{8} - \frac{3}{4} = \frac{1}{8}$ P(3) = 1 - $\frac{7}{8} = \frac{1}{8}$

x	1	2	3
$P(X = x)$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

Expected value of a discrete random variable (E(x) or μ)

Sometimes we call it the mean and it's a theoretical value that gives information about the probability distribution

$$E(X) = \sum xP(X = x)$$

Example 1:

A fair six-sided dice is rolled. The number that appears on the uppermost face is modelled by the random variable X.

a Write down the probability distribution of X.

b Use the probability distribution of X to calculate E(X).

a.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b. $E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} =$ 3.5

Example 2:

The random variable X has a probability distribution as shown in the table.

x	1	2	3	4	5
P(x)	0.1	p	0.3	q	0.2

a Given that E(X) = 3, write down two equations involving p and q.

b Find the value of p and the value of q.

a. 0.1 + p + 0.3 + q + 0.2 = 1

p + q = 0.4 (1)

1x0.1 + 2p + 3x0.3 + 4q + 5x0.2 = 3

2p + 4q =1 (2)

b. Solving equations (1) and (2) simultaneously

- 2p - 2q = - 0.8

2p + 4q = 1

2q = 0.2 q = 0.1

substituting in (2) 2p + 4(0.1) = 1 p = 0.3

Expected value of X^2

$$E(X^2) = \sum x^2 P(X = x)$$

Example 3:

A discrete random variable X has the following probability distribution:

x	1	2	3	4
$P(X = x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

a Write down the probability distribution for X^2

b Find $E(X^2)$

a.

x	1	2	3	4
x^2	1	4	9	16
$P(X = x^2)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

b. $E(X^2) = \sum x^2 P(X = x) = 1 \times \frac{12}{25} + 4 \times \frac{6}{25} + 9 \times \frac{4}{25} + 16 \times \frac{3}{25}$ = 4.8

Variance of a discrete random variable

$$Var(X) = E\left(\left(X - E(X)\right)^2\right)$$

Or

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$

N.B:

From the definition you can see that Var(X) $\geqslant 0$ for any random variable X. The variance of a discrete random variable is a measure of spread for a distribution of a random variable that determines the degree to which the values of a random variable differ from the expected value. In other words, the larger the value of Var(X), the more likely it is to take values significantly different to its expected value.

Example:

A fair six-sided dice is rolled. The number on the uppermost face is modelled by the random variable X.

Find Var(X)

x	1	2	3	4	5	6
x^2	1	4	9	16	25	36
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X^2) = \sum x^2 P(X = x) = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} + 25 \times \frac{1}{6} + 36 \times \frac{1}{6} = \frac{91}{6}$$

$$E(X) = \sum xP(X = x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \frac{91}{6} - 3.5^2 = \frac{35}{12}$$

N.B: Just to show the other method (Although it takes more time). We could have calculated $\left(X - E(X)\right)^2$.$E(X) = 3.5$ as calculated above

x	1	2	3	4	5	6
$\left(X - E(X)\right)^2$	6.25	2.25	0.25	0.25	2.25	6.25

$$Var(X) = E\left(\left(X - E(X)\right)^2\right) = \sum\left(X - E(X)\right)^2 P(X = x) = 6.25 \times \frac{2}{6} + 2.25 \times \frac{2}{6} + 0.25 \times \frac{2}{6}$$

$$= \frac{35}{12}$$

Expected value and variance of a function of X

Sometimes we use a function g(X) to simplify the main discrete random variable X

Some rules will come handy:

- $E\left(g(X)\right) = \sum g(x)P(X = x)$
- $E(aX + b) = aE(X) + b$ Where a and b are constants
- $E(X + Y) = E(X) + E(Y)$ Where X and Y are discrete variables
- $Var(aX + b) = a^2Var(X)$ Where a and b are constants

In the following examples we will understand how those rules work

Example 1:

A discrete random variable X has the probability distribution:

x	1	2	3	4
$P(X = x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

a Write down the probability distribution for Y, where $Y = 2X + 1$

b Find $E(Y)$

c Compute $E(X)$ and verify that $E(Y) = 2E(X) + 1$

a.

x	1	2	3	4
y	2(1) + 1 = 3	5	7	9
P(Y=y)	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

b. $E(Y) = \sum yP(Y = y) = 3 \times \frac{12}{25} + 5 \times \frac{6}{25} + 7 \times \frac{4}{25} + 9 \times \frac{3}{25} = 4.84$

c. $E(X) = \sum xP(X = x) = 1 \times \frac{12}{25} + 2 \times \frac{6}{25} + 3 \times \frac{4}{25} + 4 \times \frac{3}{25} = 1.92$

Computing $2E(X) + 1$ = 2(1.92) + 1 = 4.84 which proves that $E(Y) = 2E(X) + 1$

Example 2:

A random variable X has $E(X) = 4$ and $Var(X) = 3$. Find:

a $E(3X)$ **b** $E(X - 2)$

c $Var(3X)$ **d** $Var(X - 2)$

e $E(X^2)$

a. Using the rule $E(aX + b) = aE(X) + b$ then $E(3X) = 3\,E(X)$

$= 3(4) = 12$

b. Using same rule $E(X - 2) = E(X) - 2 = 4 - 2 = 2$

c. Using $Var(aX + b) = a^2Var(X)$ then $Var\ (3X) = 3^2Var(X)$

$= 9(3) = 27$

d. Using same rule $Var\ (X - 2) = Var(X) = 3$

e. $Var(X) = E(X^2) - (E(X))^2$ $3 = E(X^2) - 16$

$E(X^2) = 19$

Example 3:

Two fair 10-cent coins are tossed. The random variable X cents represents the total value of the coins that land heads up.

a Find E(X) and Var(X).

The random variables S and T are defined as follows:

$$S = X - 10 \text{ and } T = \tfrac{1}{2}X - 5$$

b Show that E(S) = E(T).

c Find Var(S) and Var (T).

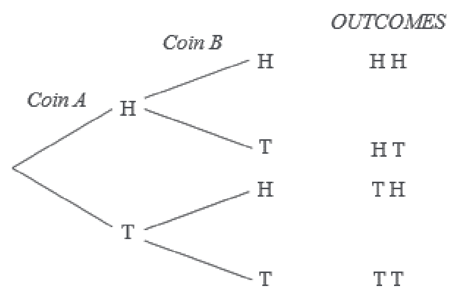
A large number of observations of S and T are taken.

d Comment on any likely differences or similarities.

Notice that the problem says value of the coins not number of heads, so when you get a head it means 10 cents. Two heads = 20 cents

a.

x	0	10	20
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$



$$E(X) = \sum xP(X = x) = 0 \times \frac{1}{4} + 10 \times \frac{1}{2} + 20 \times \frac{1}{4} = 10$$

$$E(X^2) = \sum x^2P(X = x) = 0 \times \frac{1}{4} + 100 \times \frac{1}{2} + 400 \times \frac{1}{4} = 150$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = 150 - 100 = 50$$

b. $E(aX + b) = aE(X) + b \;\therefore\; E(S) = E(X - 10) = E(X) - 10 = 10 - 10 = 0$

$\therefore E(T) = E\left(\frac{1}{2}X - 5\right) = \frac{1}{2}E(X) - 5 = 5 - 5 = 0$

c. Adding and/or subtracting does not change the variance,

so Var (S) =Var (X) = 50

$Var(aX + b) = a^2\, Var(X)$

$\therefore$ Var (T) = $\left(\frac{1}{2}\right)^2 Var(X) = \frac{1}{4} \times 50 = 12.5$

d. As more observations are taken, we should be closer to the theoretical values of the expectation of each which is zero. The spread of S will still be more than that of T

Example 4:

The random variable X has the following probability distribution:

x	0°	30°	60°	90°
$\mathbf{P}(X = x)$	0.4	0.2	0.1	0.3

Calculate E(sin X).

sin(0) = 0, sin(30) = 0.5, sin(60) = $\frac{\sqrt{3}}{2}$, sin(90) = 1

$E\big(g(X)\big) = \sum g(x) P(X = x) \therefore E(\sin X) = \sin \sum sin(x)\, P(X = x) =$

$= 0 \times 0.4 + 0.5 \times 0.2 + \frac{\sqrt{3}}{2} \times 0.1 + 1 \times 0.3 = 0.487$ (3 s.f.)

Example 5:

X is a discrete random variable. The discrete random variable Y is defined as $Y = \frac{X - 150}{50}$

Given that E(Y) = 5.1 and Var(Y) = 2.5, find:

a E(X)

b Var(X).

a. $Y = \frac{X - 150}{50}$ $\therefore X = 50\, Y + 150$

$E(X) = E(50Y + 150) = 50E(Y) + 150 = 50 \times 5.1 + 150 = 405$

b. $Var(X) = Var(50Y + 150) = 50^2 Var(Y) = 2500 \times 2.5 = 6250$

Special Rules:

Discrete *Uniform* Distribution: Defined over {1, 2, 3, …, n}

and all probabilities are the same $P(X = x) = \frac{1}{n}$

Example: **The probability distribution for the score S on a single roll of a dice is:**

s	1	2	3	4	5	6
$P(S = s)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \frac{n+1}{2}$$

$$Var(X) = \frac{(n+1)(n-1}{12}$$

Example

Digits are selected at random from a table of random numbers.

a Find the mean and standard deviation of a single digit.

b Find the probability that a particular digit lies within one standard deviation of the mean.

a. Digits can be {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} only so 10 digits

So let's call this random variable R

But we know that the random variable X has to take values from 1 to n

so R = X – 1 and thus X takes values from 1 to 10 (n = 10)

$E(R) = E(X-1) = E(X) - 1 = \frac{n+1}{2} - 1 = \frac{10+1}{2} - 1 = 4.5$

$Var(R) = Var(X-1) = Var(X) = \frac{(n+1)(n-1)}{12} = \frac{(10+1)(10-1)}{12} = 8.25$

$\sigma = \sqrt{8.25} = 2.87$

b. $P(M - \sigma < R < M + \sigma) = P(4.5 - 2.87 < R < 4.5 + 2.87) = P(1.63 < R < 7.37)$

Rounding, because digits have no decimals $P(2 < R < 7) = \frac{6}{10} = \frac{3}{5}$