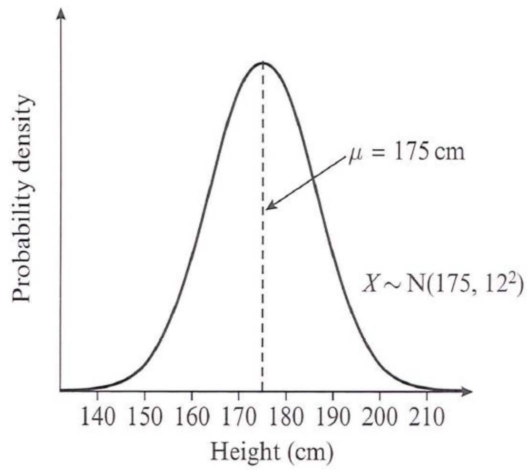
The Normal Distribution:



A continuous random variable can take any one from an unlimited number of values.

The probability of taking one specific value is zero but it usually refers to a range and is drawn as a curve.

The curve has specific characteristics

- It looks like a bell (Bell curve)
- It is symmetric about the population mean μ
- Total area under the curve is 1 (The total probability is 1)
- It has points of inflection at $\mu + \sigma$ and $\mu - \sigma$
- Approximately 68% of the data lies within one standard deviation of the mean
- 95% of the data lies within two standard deviations of the mean
- 99.7% (Almost all) of the data lies within three standard deviations of the mean

Important notation: If X is a normally distributed random variable we write

$X \sim N(\mu, \sigma^2)$

Example 1:

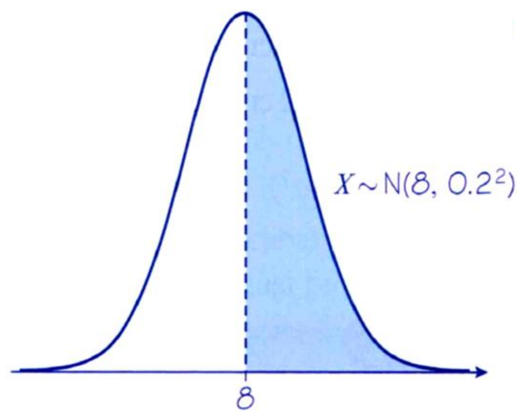
The diameters of a metal pin produced by a particular machine, X mm, are modelled as $X \sim N(8, 0.2^2)$. Find:

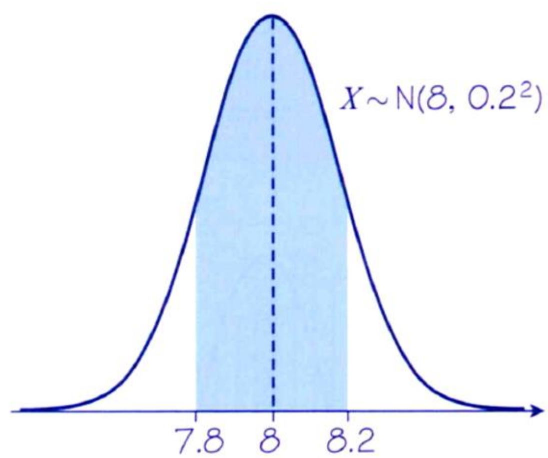
a $P(X > 8)$

b $P(7.8 < X < 8.2)$

a. $\mu = 8$ so $P(X > 8)$ refers to the right half of the curve. We know that the whole area =1 so one half is 0.5

$P(X > 8)$ = 0.5



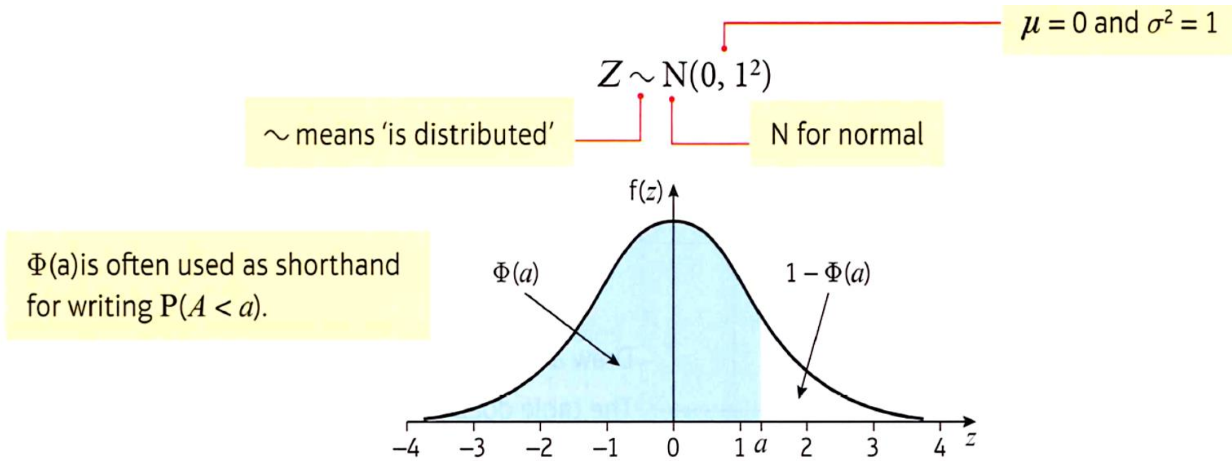


b. $\sigma = 0.2$ so 8 – 0.2 = 7.8, 8 + 0.2 = 8.2 therefore we actually have

$P(7.8 < X < 8.2) = P(\mu - \sigma < X < \mu + \sigma)$

= 0.68

The Standard Normal Distribution



The Normal Distribution Function

The function tabulated below is $\Phi(z)$, defined as $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} \, dt$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.9772
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.02	0.9783
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.04	0.9793
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.06	0.9803
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.08	0.9812
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.10	0.9821
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.12	0.9830
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.14	0.9838
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.16	0.9846
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.18	0.9854
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.20	0.9861
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.22	0.9868
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.24	0.9875
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.26	0.9881
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.28	0.9887
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.30	0.9893
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.32	0.9898
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.34	0.9904
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.36	0.9909
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.38	0.9913
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.40	0.9918
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.42	0.9922
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.44	0.9927
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.46	0.9931
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.48	0.9934
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.50	0.9938
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.55	0.9946
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.60	0.9953
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.65	0.9960
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.70	0.9965
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.75	0.9970
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.80	0.9974
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.85	0.9978
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.90	0.9981
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.95	0.9984
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	3.00	0.9987

0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	3.05	0.9989
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	3.10	0.9990
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	3.15	0.9992
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	3.20	0.9993
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	3.25	0.9994
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	3.30	0.9995
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	3.35	0.9996
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	3.40	0.9997
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	3.50	0.9998
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	3.60	0.9998
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	3.70	0.9999
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	3.80	0.9999
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	3.90	1.0000
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	4.00	1.0000
0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.9772		

Percentage Points Of The Normal Distribution

The values z in the table are those which a random variable $Z \sim N(0, 1)$ exceeds with probability p; that is, $P(Z > z) = 1 - \Phi(z) = p$.

p	z	p	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

Example 2:

Use the normal distribution tables to find:

a $P(Z < 1.54)$

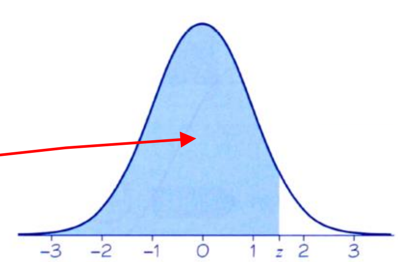
b $P(Z > 2.65)$

c $P(Z < -0.75)$

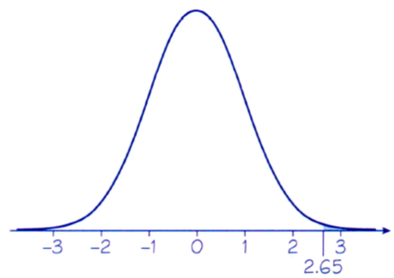
d $P(-1.20 < Z < 1.40)$

a. $P(Z < 1.54) = 0.9382$ from the table

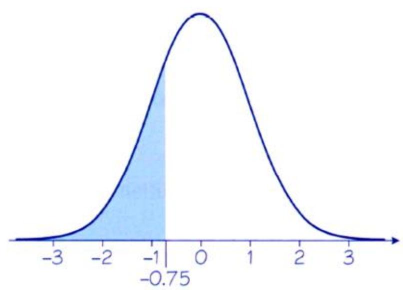
Shade the required area, then use the table



b. $P(Z > 2.65) = 1 - P(Z < 2.65)$ = 1 - 0.9960 = 0.004



c. $P(Z < -0.75) = P(Z > 0.75) = 1 - P(Z < 0.75)$ = 1 – 0.7734 = 0.2266

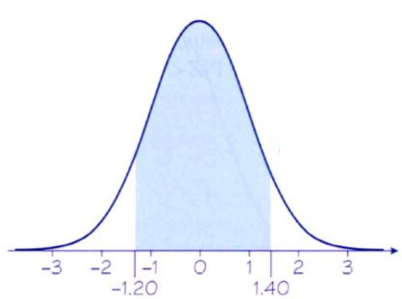


d. $P(-1.20 < Z < 1.40) = P(Z < 1.40) - P(Z < -1.20)$

$P(Z < 1.40) = 0.9192$ from the table

$P(Z < -1.20) = P(Z > 1.20) = 1 - P(Z < 1.20)$ = 1 – 0.8849 =0.1151

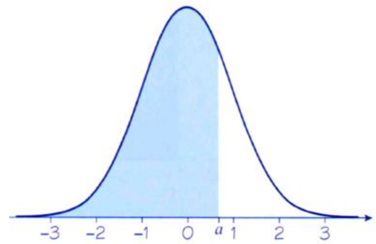
$P(-1.20 < Z < 1.40) = 0.9192 - 0.1151 = 0.8041$



Example 3:

Find the value of the constant a such that $P(Z < a) = 0.7517$

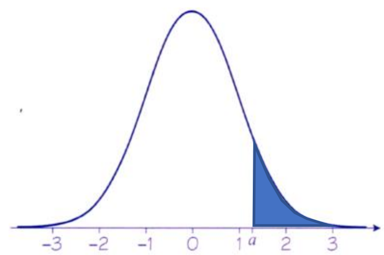
Draw a diagram to help visualise the problem
look in the table for an area of 0.7517
you will find that a = 0.68



Example 4:

Find the value of the constant a such that $P(Z > a) = 0.100$

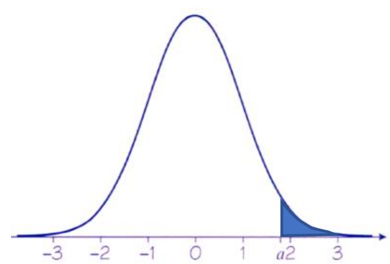
For $P(Z > a)$ it's better to check the percentage points table for the given value it makes solving a lot faster.
In this problem 0.100 is listed and a = 1.2816



Example 5:

Find the value of the constant a such that $P(Z > a) = 0.0322$

Although this is exactly like the previous example, yet unfortunately we can't find 0.0322 in the percentage points table.
We have to solve it the usual way then.
Find 1 – 0.0322 = 0.9678
now $P(Z < a) = 0.9678$ which gives a = 1.85



Example 6:

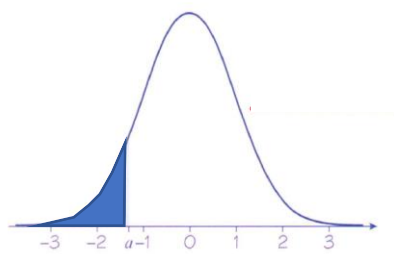
Find the value of the constant a such that $P(Z < a) = 0.1075$

First we draw the problem

0.1075 is not in the table so we use symmetry

1-0.1075 = 0.8925 which gives a = 1.24

so our answer is a = - 1.24



<u>Example 7:</u>

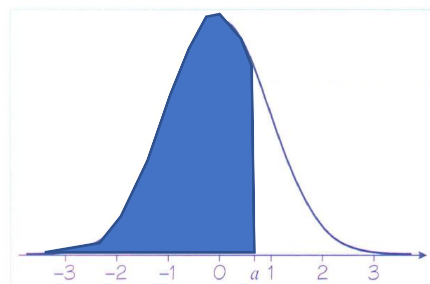
Use the tables to find $P(Z < a) = 0.75$

0.75 is not an exact number in the table

our options are either 0.7486 or 0.7517

where 0.7486 is closer so we use it to get

a = 0.67



<u>Standard Normal distribution:</u>

The data is coded so we get a mean of 0 and standard deviation of 1. This gives us the ability to use the standard table.

To do this we change Z to

$$Z = \frac{X - \mu}{\sigma}$$

Example 1:

The random variable $X \sim N(50, 4^2)$. Find:

a $P(X < 53)$ **b** $P(X \geqslant 55)$

a. $Z = \frac{X-\mu}{\sigma} = \frac{53-50}{4} = 0.75$ $P(X < 53) = P(Z < 0.75)$ = 0.7734

b. $Z = \frac{X-\mu}{\sigma} = \frac{55-50}{4} = 1.25$ $P(X > 55) = P(Z > 1.25)$

The table says that $P(Z < 1.25) = 0.8944$

so $P(X > 55) = P(Z > 1.25) = 1 - 0.8944 = 0.1056$

Example 2:

The random variable $Y \sim N(20, 9)$.

Find the value of b such that $P(Y > b) = 0.0485$

$P(Y > b) = 0.0485 \; so \; P(Y < b) = 1 - 0.0485 = 0.9515$

$P\left(Z < \frac{b-20}{3}\right)$ = 0.9515 From the table 0.9515 gives a = 1.66

$\frac{b-20}{3} = 1.66$ so b = 24.98

Example 3:

The random variable $X \sim N(\mu, 3^2)$.

Given that $P(X > 20) = 0.20$, find the value of μ.

This one is in the percentage points table 0.2 gives a = 0.8416 so no need to search in the standard normal distribution table

$\frac{20-\mu}{3} = 0.8416$ $\quad\quad$ $\mu = 17.5$ (3 s.f.)

Example 4:

A machine makes metal sheets with width, X cm, modelled as a normal distribution such that $X \sim N(50, \sigma^2)$.

a Given that $P(X < 46) = 0.2119$, find the value of σ.

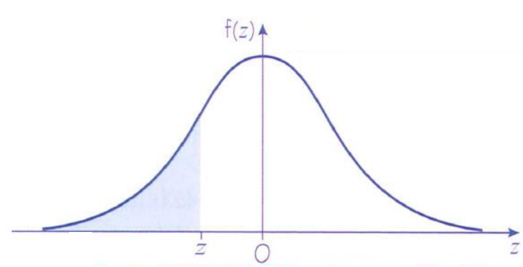
b Find the 90th percentile of the widths.

a. Using symmetry, we look for

$1 - 0.2119 = 0.7881$

then a = - 0.8

$\frac{46-50}{\sigma} = -0.8$ $\quad\quad$ so $\sigma = 5$



b. 90^{th} percentile = 0.9

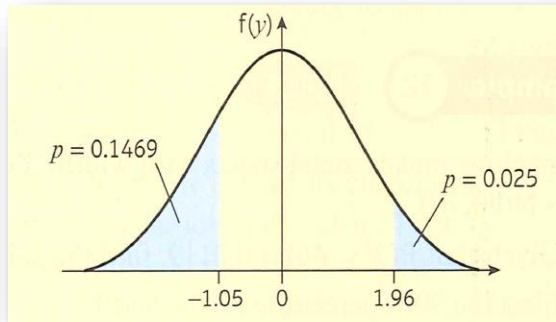
$P(X < a) = 0.9$ $\quad$ $P(X > a) = 0.1$ $\quad$ Using the percentage points table

a = 1.2816

so $\frac{x-50}{5} = 1.2816$ $\quad\quad$ 90^{th} percentile of the widths = 56.4

Example 5:

The random variable $X \sim N(\mu, \sigma^2)$.
Given that $P(X > 35) = 0.025$ and $P(X < 15) = 0.1469$, find the value of μ and the value of σ.



First, we draw the problem

The right part $P(Z > a_1) = 0.025$

can be found in the percentage points

table a_1 = 1.96

The left part we use symmetry

$P(Z < a_2) = 0.1469$ so we look in the standard normal distribution table

for 1- 0.1469 = 0.8531 which gives 1.05 so a_2 = - 1.05

Now we standardize our data $\frac{35-\mu}{\sigma}$ =1.96, $1.96\,\sigma + \mu = 35$ .... (1)

$\frac{15-\mu}{\sigma}$ = - 1.05 , $-1.05\,\sigma + \mu$ =15 .... (2)

Solving for equations (1) and (2) simultaneously we get

σ = 6.64 (3 s.f.) μ = 22.0 (3 s.f.)